Relational semantics for a fragment of linear logic

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March 4, 2011

Abstract

Relational semantics, given by Kripke frames, play an essential role in the study of modal and intuitionistic logic. In [4] it is shown that the theory of relational semantics is also available in the more general setting of substructural logic, at least in an algebraic guise. Building on these ideas, in [5] a type of frames is described which generalize Kripke frames and provide semantics for substructural logics in a purely relational form.

We will extend the work in [4, 5] and use their approach to obtain relational semantics for multiplicative additive linear logic. Hereby we illustrate the strength of using canonical extensions to retrieve relational semantics: it allows a modular and uniform treatment of additional operations and axioms.

Traditionally, so-called phase spaces are used to describe semantics for linear logic [8]. These have the drawback that, contrary to our approach, they do not allow a modular treatment of additional axioms. However, the two approaches are related, as we will explain.

1 Introduction

Relational semantics, given by Kripke frames, play an essential role in the study of modal and intuitionistic logic [3]. They provide an intuitive interpretation of the logic and a means to obtain information about it. The possibility of applying semantical techniques to obtain information about a logic motivates the search for relational semantics in a more general setting.

Many logics are closely related to corresponding classes of algebraic structures which provide *algebraic semantics* for the logics. The algebras associated to classical modal logic are Boolean algebras with an additional operator (BAOs). Kripke frames arise naturally from the duality theory for these structures in the following way. Boolean algebras are dually equivalent to Stone spaces [11]. A modal operator on Boolean algebras translates to a binary relation with certain topological properties on the corresponding dual spaces, hence giving rise to so-called descriptive general frames. Forgetting the topology yields

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Kripke frames, which are in a discrete duality with complex modal algebras, *i.e.*, modal algebras whose underlying Boolean algebra is a powerset algebra. This may be depicted as follows:

$$
\begin{tikzcd}
\text{syntactic specification} \arrow[d, leftrightsquigarrow] & \\
\text{modal algebras} \arrow[r, shift left] \arrow[d, "\text{canonical extension}"'] & \text{descriptive general frames} \arrow[l, shift left, "\text{topological duality}"] \arrow[d, "\text{forget topology}"] \\
\text{complex modal algebras} \arrow[r, shift left] & \text{Kripke frames} \arrow[l, shift left, "\text{discrete duality}"] \arrow[d, leftrightsquigarrow] \\
& \text{relational semantics}
\end{tikzcd}
$$

Hence, one may retrieve relational semantics for modal logic by first moving horizontally using the duality and thereafter going down by forgetting the topology.

Many other interesting logics, including substructural logics, however, have algebraic semantics which are not based on distributive lattices and for these duality theory is vastly more complicated or even non-existent. Luckily, the picture above also indicates an alternative route to obtain relational semantics: going down first and thereafter going right. The (left) downward mapping is given by taking the *canonical extension* of a BAO. Canonical extensions were introduced in the 1950s by Jónsson and Tarski exactly for BAOs [9, 10]. Thereafter their ideas have been developed further, which has led to a smooth theory of canonical extensions applicable in a broad setting [6, 7]. In [4] canonical extensions of partially ordered algebras are defined to obtain relational semantics for the fusion-implication fragment of various substructural logics. Their approach is purely algebraic. In [5] this work is translated to the setting of possible world semantics. A class of frames (RS-frames) is described which generalize Kripke frames and provide semantics for substructural logics in a purely relational form. This is summarized in the following picture:

$$
\begin{tikzcd}
\text{syntactic specification} \arrow[d, leftrightsquigarrow] & \\
\text{partially ordered algebras} \arrow[d, "\text{canonical extension}"'] & \\
\text{perfect lattices with add'al operations} \arrow[r, shift left] & \text{RS-frames with add'al relations} \arrow[l, shift left, "\text{discrete duality}"] \arrow[d, leftrightsquigarrow] \\
& \text{relational semantics}
\end{tikzcd}
\tag{1}
$$

A well-know substructural logic is linear logic. Linear logic was introduced by Jean-Yves Girard [8]. Formulas in linear logic represent resources which may be used exactly once. Proof-theoretically this is witnessed by the fact that the structural rules contraction and weakening are not admissible in general. However, these structural rules are allowed in a controlled way by means of a new modality, the exponential !, which expresses the case of unlimited availability of a specific resource. Traditionally, phase spaces are used as semantics for linear logic. These have the drawback that, contrary to the approach described above, they do not allow a modular treatment of additional operations and axioms.

In this paper we discuss part of a joint project with Mai Gehrke and Lorijn van Rooijen on developing relational semantics for full linear logic. We focus on multiplicative additive linear logic (MALL), the fragment of linear logic that leaves the exponential out of consideration, and describe how to obtain relational semantics for MALL. Thereby we illustrate that canonical extensions allow a modular and uniform treatment of additional operations and axioms, which distinguishes our work from earlier derivations of Kripke-style semantics for linear logic [1].

First, we discuss the general method of obtaining relational semantics for substructural logics using canonical extension, essentially by explaining how to move 'down-right' in the picture above (Section 2) and by indicating how to show that this indeed yields complete relational semantics (Section 3). We focus on the parts of this general theory that are important for the remainder of our paper and refer the reader to [4, 5] for more details. In Section 4 this method is applied to obtain relational semantics for MALL. Finally, in Section 5 we discuss how our results relate to phase spaces.

2 Duality between perfect lattices and RS-polarities

Algebraic semantics for substructural logics are given by partially ordered sets (posets) with additional operations on them (*partially ordered algebras*). Hence, the first step in obtaining relational semantics for substructural logics using the method depicted in Figure 1 is to define canonical extensions of posets. This is worked out in Section 2 of [4] where one can find a careful and clear explanation of this theory. The structures arising as canonical extensions of posets are perfect lattices.

Definition 1 *A* perfect lattice *is a complete lattice that is both join-generated by its completely join-irreducible elements and meet-generated by its completely meet-irreducible elements.*

To move horizontally in Figure 1 one should identify relational structures that are in a duality with perfect lattices. In [5] a class of (two-sorted) frames fulfilling this requirement is described. We briefly discuss this duality.

Definition 2 *A (two-sorted) frame is a triple $F = (X, Y, \preccurlyeq)$ where X and Y are sets and $\preccurlyeq \subseteq X \times Y$ is a relation from X to Y.*

A frame gives rise to a Galois connection between $\wp(X)$ and $\wp(Y)$:

$$\begin{array}{rclcl} (\ )^u & : & \wp(X) & \to & \wp(Y) \\ & & A & \mapsto & \{y \in Y \mid \forall x.\ x \in A \Rightarrow x \preccurlyeq y\} \end{array}$$

$$\begin{array}{rclcl} (\ )^l & : & \wp(Y) & \Rightarrow & \wp(X) \\ & & B & \mapsto & \{x \in X \mid \forall y.\ y \in B \Rightarrow x \preccurlyeq y\} \end{array}$$

The complete lattice of Galois closed subsets of X is given by

$$\mathcal{G}(F) \ = \ \{A \subseteq X \mid (A^u)^l = A\}$$

which is a perfect lattice.

Conversely, for every perfect lattice $\mathbf{L}$, we define a frame $\mathcal{F}(\mathbf{L})$ by $X = \mathcal{J}^\infty(\mathbf{L})$, $Y = \mathcal{M}^\infty(\mathbf{L})$ and, for all $x \in X, y \in Y$,

$$x \preccurlyeq y \quad \Leftrightarrow \quad x \leq_{\mathbf{L}} y.$$

This frame is *separating*, i.e., the following two conditions hold:

1. $\forall x_1, x_2 \in X\ (x_1 \neq x_2 \Rightarrow \{x_1\}^u \neq \{x_2\}^u)$;
2. $\forall y_1, y_2 \in Y\ (y_1 \neq y_2 \Rightarrow \{y_1\}^l \neq \{y_2\}^l)$.

Furthermore it is *reduced*, i.e., the following two conditions hold:

1. $\forall x \in X \exists y \in Y\ (x \not\preccurlyeq y$ and $\forall x' \in X\ [\{x'\}^u \supset \{x\}^u \Rightarrow x' \preccurlyeq y])$,
2. $\forall y \in Y \exists x \in X\ (y \not\succcurlyeq x$ and $\forall y' \in Y\ [\{y'\}^l \subset \{y\}^l \Rightarrow y' \succcurlyeq x])$.

A frame that is both separating and reduced is called an *RS-frame*. The separating property implies that the maps

$$\begin{array}{rcl} X & \to & \mathcal{G}(F) \\ x & \mapsto & (\{x\}^u)^l \end{array} \qquad\qquad \begin{array}{rcl} Y & \to & \mathcal{G}(F) \\ y & \mapsto & \{y\}^l \end{array}$$

are injective. Therefore we may think of X and Y as subsets of $\mathcal{G}(F)$ and we will write x both for the element of X and for the corresponding element $\{x\}^{ul}$ of $\mathcal{G}$(F) (and similarly for elements of Y). For an S-frame, being reduced exactly means that all elements of X are completely join-irreducible in $\mathcal{G}(F)$ and the elements of Y are completely meet-irreducible in $\mathcal{G}(F)$.

An RS-frame morphism $F_1 = (X_1, Y_1, \leq) \to (X_2, Y_2, \leq) = F_2$ is a pair of relations $S_1 \subseteq Y_1 \times X_2$, $S_2 \subseteq X_1 \times Y_2$ satisfying some conditions. These conditions ensure that the pair of relations gives rise to a complete lattice homomorphism $\mathcal{G}(S_1, S_2) \colon \mathcal{G}(F_2) \to \mathcal{G}(F_1)$. Conversely, for each complete lattice homomorphism $f \colon \mathbf{L_1} \to \mathbf{L_2}$ between perfect lattices, one may define an RS-frame morphism $\mathcal{F}(f) \colon \mathcal{F}(\mathbf{L}_2) \to \mathcal{F}(\mathbf{L}_1)$.

Proposition 3 *The mappings $\mathcal{F}$ and $\mathcal{G}$ form a duality between the category of perfect lattices and the category of RS-frames.*

For further details and a proof of the above proposition, the reader is referred to [5].

3 Relational semantics via canonical extension

We will now extend and apply the basic theory of the previous section to describe the general method for obtaining relational semantics for substructural logics.

The basic substructural logic we consider is non-associative Lambek calculus (NLC). Its signature consists of three binary operations $\otimes$, $\rightarrow$, $\leftarrow$. The axioms of NLC state that the implications $\rightarrow$ and $\leftarrow$ are residuals of the fusion $\otimes$. Algebraic semantics for this logic are given by residuated algebras.

Definition 4 *A* residuated algebra *is a structure* $(\mathbf{P}, \otimes, \rightarrow, \leftarrow)$*, where* $\mathbf{P}$ *is a partially ordered set and, for all* $x, y, z \in \mathbf{P}$*,*

$$\begin{array}{rcl} x \otimes y \leq z & \Leftrightarrow & y \leq x \rightarrow z \\ & \Leftrightarrow & x \leq z \leftarrow y. \end{array}$$

A residuated algebra is called perfect *if its underlying poset is a perfect lattice.*

For a perfect residuated algebra, the underlying perfect lattice $\mathbf{L}$ corresponds dually to the RS-frame $\mathcal{F}(\mathbf{L}) = (\mathcal{J}^\infty(\mathbf{L}), \mathcal{M}^\infty(\mathbf{L}), \leq_{\mathbf{L}})$, as explained in Section 2. The action of the fusion (and thereby of its residuals) may be encoded on this dual frame as follows. First note that, as the fusion is residuated, it is completely join preserving in both coordinates. Therefore, its action is completely determined by its action on pairs from $J^\infty(\mathbf{L}) \times J^\infty(\mathbf{L})$. Define a relation $R_\otimes \subseteq X \times X \times Y$ by

$$R_\otimes(x_1, x_2, y) \Leftrightarrow x_1 \otimes x_2 \leq y.$$

The relation $R_\otimes$ is *compatible*, that is, for all $x_1, x_2 \in X$, $y \in Y$, the sets

$$R_\otimes[x_1, x_2, \_] \qquad R_\otimes[x_1, \_, y] \qquad R_\otimes[\_, x_2, y]$$

are Galois closed. [1]

Definition 5 *A structure* $F = (X, Y, \preccurlyeq, R)$*, where* $(X, Y, \preccurlyeq)$ *is an RS-frame and* $R \subseteq X \times X \times Y$ *is a compatible relation, is called a* relational RS-frame*.*

Conversely, for an RS-frame $F = (X, Y, \preccurlyeq)$, a relation $R \subseteq X \times X \times Y$ gives rise to a fusion $\otimes_R$ on $\mathcal{G}(F)$, by defining

$$\begin{array}{rcll} x_1 \otimes_R x_2 & = & \bigwedge\{y \in Y \mid R(x_1, x_2, y)\} & \text{for all } x_1, x_2 \in X, \\ w_1 \otimes_R w_2 & = & \bigvee\{x_1 \otimes_R x_2 \mid x_1 \leq u_1,\ x_2 \leq u_2\} & \text{for all } w_1, w_2 \in \mathcal{G}(F). \end{array}$$

This operation is completely join preserving in both coordinates and therefore it is residuated, with residuals $\rightarrow_R$ and $\leftarrow_R$.

For any residuated fusion operation $\otimes$ on a perfect lattice, $\otimes_{R_\otimes} = \otimes$ and, for any compatible relation R on an RS-frame, $R_{\otimes_R} = R$.

[1]We may also witness the fusion $\otimes$ dually by the relation $R_\downarrow \subseteq X^3$ defined by $R_\downarrow(x_1, x_2, x_3) \Leftrightarrow x_3 \leq x_1 \otimes x_2$. In that case, however, the conditions stating that R arises from a fusion are less natural.

Proposition 6 (Proposition 6.6 in [4]) *The above defined maps* $(\mathbf{L}, \otimes, \to, \leftarrow) \mapsto (\mathcal{F}(L), R_{\otimes})$ *and* $(X, Y, \preccurlyeq, R) \mapsto (\mathcal{G}(X, Y, \preccurlyeq), \otimes_R, \to_R, \leftarrow_R)$ *yield a duality between perfect residuated algebras and relational RS-frames.*[2]

In the remainder we will also denote the extended mappings of the above proposition (and any further generalizations) by $\mathcal{F}$ and $\mathcal{G}$.

One may define a satisfaction relation $\Vdash$ on relational frames, such that, for all frames F, for all formulas ϕ, ψ in NLC,

$$\phi \Vdash \psi \ \text{ holds in } F \qquad \Leftrightarrow \qquad \phi \leq \psi \ \text{ holds in } \mathcal{G}(F). \tag{2}$$

This is explained in detail in Section 4 of [5].

For a residuated algebra $\mathbf{P}$, the σ-extension of the fusion on $\mathbf{P}$, $\otimes^{\sigma} \colon \mathbf{P}^{\delta} \times \mathbf{P}^{\delta} \to \mathbf{P}^{\delta}$, is a residuated operator on the canonical extension $\mathbf{P}^{\delta}$ (Corollary 3.7 of [4]). This completes the description of the walk through Figure 1 for NLC: We start with an residuated algebra $\mathbf{P}$, its canonical extension is a perfect residuated algebra $\mathbf{P}^{\delta}$ which gives rise to a relational frame via the mapping $\mathcal{F}$.

We are now ready to describe our method for obtaining relational semantics for a substructural logic. Let $\mathcal{E}$ be a collection of inequalities axiomatizing a logic $\mathcal{L}_{\mathcal{E}}$ in the connectives $\otimes, \to, \leftarrow$, extending NLC. The collection $\mathcal{A}lg_{\mathcal{E}}$ of residuated algebras satisfying the inequalities in $\mathcal{E}$ provides complete algebraic semantics for $\mathcal{L}_{\mathcal{E}}$, in the sense that, for all formulas ϕ, ψ,

$$\phi \vdash \psi \text{ is derivable in } \mathcal{L}_{\mathcal{E}} \ \text{ iff } \ \phi \leq \psi \text{ holds in all residuated algebras in } \mathcal{A}lg_{\mathcal{E}}.$$

Our aim is to describe a collection of relational frames $\mathcal{K}$ which provides complete relational semantics for $\mathcal{L}_{\mathcal{E}}$. We define, for a collection of relational frames $\mathcal{K}$,

$$\mathcal{K}^{+} = \{\mathcal{G}(F) \mid F \in \mathcal{K}\}.$$

By (2), $\mathcal{K}$ provides complete relational semantics for $\mathcal{L}_{\mathcal{E}}$ iff $\mathcal{L}_{\mathcal{E}} = \mathcal{E}qThr(\mathcal{K}^{+})$, where $\mathcal{E}qThr(\mathcal{K}^{+})$ is the equational theory of $\mathcal{K}^{+}$, *i.e.* the collection of inequalities that hold in all algebras in $\mathcal{K}^{+}$.

To obtain complete relational semantics for $\mathcal{L}_{\mathcal{E}}$ it suffices to obtain:

1. Canonicity: show that $\mathcal{A}lg_{\mathcal{E}}$ is closed under canonical extension, that is, show that, for all $\mathbf{P} \in \mathcal{A}lg_{\mathcal{E}}$, $\mathbf{P}^{\delta} \in \mathcal{A}lg_{\mathcal{E}}$.

2. Correspondence: give necessary and sufficient conditions on relational frames F to ensure that $\mathcal{G}(F) \in \mathcal{A}lg_{\mathcal{E}}$.

Proposition 7 *If* $\mathcal{A}lg_{\mathcal{E}}$ *is closed under canonical extension, then* $\mathcal{E}qThr(\mathcal{A}lg_{\mathcal{E}}) = \mathcal{E}qThr(\mathcal{A}lg_{\mathcal{E}}^{\delta})$*, where* $\mathcal{A}lg_{\mathcal{E}}^{\delta} = \{\mathbf{P}^{\delta} \mid \mathbf{P} \in \mathcal{A}lg_{\mathcal{E}}\}$.

Proof As, by assumption, $\mathcal{A}lg_{\mathcal{E}}^{\delta} \subseteq \mathcal{A}lg_{\mathcal{E}}$, clearly $\mathcal{E}qThr(\mathcal{A}lg_{\mathcal{E}}) \subseteq \mathcal{E}qThr(\mathcal{A}lg_{\mathcal{E}}^{\delta})$. For the converse, suppose $\phi \leq \psi$ holds in $\mathcal{A}lg_{\mathcal{E}}^{\delta}$ and $\mathbf{P} \in \mathcal{A}lg_{\mathcal{E}}$. As $\mathbf{P}$ embeds in its canonical extension $\mathbf{P}^{\delta}$ and $\mathbf{P}^{\delta} \in \mathcal{A}lg_{\mathcal{E}}^{\delta}$, $\phi \leq \psi$ holds in $\mathbf{P}$. $\square$

[2] Note that we have not spelled out which morphisms we consider in both categories. The reader interested in more details is referred to [4].

If $\mathcal{A}lg_{\mathcal{E}}$ is closed under canonical extension we say the collection of axioms $\mathcal{E}$ is *canonical*. It follows from the above proposition that in this case the collection

$$\mathcal{K} = \{F \mid \mathcal{G}(F) \in \mathcal{A}lg_{\mathcal{E}}\}$$

provides complete relational semantics for $\mathcal{L}_{\mathcal{E}}$ (note that $\mathcal{A}lg_{\mathcal{E}}^{\delta} \subseteq \mathcal{K}^{+} \subseteq \mathcal{A}lg_{\mathcal{E}}$). In case the axioms in $\mathcal{E}$ are 'sufficiently simple' one may obtain, in a mechanical way, first order conditions on relational frames F that are necessary and sufficient to ensure $\mathcal{G}(F) \in \mathcal{A}lg_{\mathcal{E}}$. Many well-known logics may be axiomatized by canonical and 'sufficiently simple' axioms, whence the above described procedure may be applied to obtain complete relational semantics. In [4] this is worked out for the fusion-implication fragment of Lambeck calculus, linear logic, relevance logic, BCK logic and intuitionistic logic.

In case the logic is equipped with additional function symbols an extension of the above method may be applied. Algebraic semantics are given by residuated algebras equipped with additional operations (corresponding to the additional function symbols). To obtain relational semantics for the logic one has to give a description of these additional operations on relational frames. In the next section we will illustrate this procedure by deriving relational semantics for multiplicative additive linear logic.

4 Relational semantics for MALL

To derive relational semantics for multiplicative additive linear logic (MALL), we start by describing its algebraic semantics. These are given by classical linear algebras, which are extensions of the residuated algebras studied in the previous section.

Definition 8 *A* classical linear algebra *(CL-algebra) is a structure* $(\mathbf{L}, \otimes, \rightarrow, \leftarrow, 1, 0)$*, where*

1. $(\mathbf{L}, \otimes, \rightarrow, \leftarrow)$ *is a residuated algebra;*
2. *the fusion* $\otimes$ *is associative and commutative and has a unit* 1*;*
3. $\mathbf{L}$ *is a bounded lattice;*
4. *for all* $a \in \mathbf{L}$*,* $(a \rightarrow 0) \rightarrow 0 = a$*.*

In linear logic, the meet operation is denoted by & (with unit $\top$), the join by $\oplus$ (with unit 0), the implication by $\multimap$ and our constant 0 is denoted by $\perp$. However, as we will refer to the literature from lattice theory we will stick to the usual lattice theoretic notation and denote meet by $\wedge$ (with unit $\top$) and join by $\vee$ (with unit $\perp$). For further details on CL-algebras the reader is referred to [12], which uses a notation similar to ours.

We will denote $x \rightarrow 0$ by $x^{\perp}$ and call this operation *linear negation*. Implication sends joins in the first coordinate to meets, hence $(\_)^{\perp}$ sends joins to meets.

As $(\_)^{\perp}$ is a bijection, it follows that it is a (bijective) lattice homomorphism $\mathbf{L} \to \mathbf{L}^{\partial}$, where $\mathbf{L}^{\partial}$ is the lattice obtained by reversing the order in $\mathbf{L}$.

The first step in obtaining relational semantics for MALL is checking canonicity, *i.e.*, ensuring that the class **CL** of CL-algebras is closed under canonical extension.

Proposition 9 *The class* **CL** *is closed under canonical extension.*

Proof Let $\mathbf{L}$ be a CL-algebra and let $\mathbf{L}^{\delta}$ be its canonical extension. In [4] it is shown that $\mathbf{L}^{\delta}$ is a perfect residuated algebra. Hence, in particular, it is a bounded lattice. Furthermore, it is shown that, if $\otimes$ is associative (resp. commutative), then so is its extension $\otimes^{\sigma}$.

It is left to show that, for all $w \in \mathbf{L}^{\delta}$, $(w^{\perp^{\delta}})^{\perp^{\delta}} = w$. This may be derived from the results in [2], but we choose to give a direct proof here to illustrate the methods of the theory of canonical extensions. As $(\_)^{\perp}$ is a lattice homomorphism $\mathbf{L} \to \mathbf{L}^{\partial}$, its extension is a complete lattice homomorphism $\mathbf{L}^{\delta} \to (\mathbf{L}^{\partial})^{\delta} = (\mathbf{L}^{\delta})^{\partial}$. Every element of the canonical extension may be written as a join of meets of elements of the original lattice. We write $K(\mathbf{L}^{\delta})$ for the elements of $\mathbf{L}^{\delta}$ that may be obtained as a meet of elements of $\mathbf{L}$. For $w \in \mathbf{L}^{\delta}$,

$$\begin{array}{rcl} w & = & \bigvee\{x \in K(\mathbf{L}^{\delta}) \mid x \leq w\} \\ & = & \bigvee\{\bigwedge\{a \in \mathbf{L} \mid x \leq a\} \mid x \in K(\mathbf{L}^{\delta}),\, x \leq w\}. \end{array}$$

Hence, for $w \in \mathbf{L}^{\delta}$,

$$\begin{array}{rcl} (w^{\perp^{\delta}})^{\perp^{\delta}} & = & ((\bigvee\{\bigwedge\{a \in \mathbf{L} \mid x \leq a\} \mid x \in K(\mathbf{L}^{\delta}),\, x \leq w\})^{\perp^{\delta}})^{\perp^{\delta}} \\ & = & (\bigwedge\{\bigvee\{a^{\perp} \mid a \in \mathbf{L},\, x \leq a\} \mid x \in K(\mathbf{L}^{\delta}),\, x \leq w\})^{\perp^{\delta}} \\ & = & \bigvee\{\bigwedge\{(a^{\perp})^{\perp} \mid a \in \mathbf{L},\, x \leq a\} \mid x \in K(\mathbf{L}^{\delta}),\, x \leq w\} \\ & = & \bigvee\{\bigwedge\{a \in \mathbf{L} \mid x \leq a\} \mid x \in K(\mathbf{L}^{\delta}),\, x \leq w\} \\ & = & w, \end{array}$$

which proves the claim. □

To describe the constants 1 and 0 dually, we have to extend the relational frames with two Galois-closed subsets, $U \subseteq X$ and $Z \subseteq Y$. Starting from a perfect CL-algebra $\mathbf{L}$, these sets are given by

$$U = \{x \in \mathcal{J}^{\infty}(\mathbf{L}) \mid x \leq 1\} \quad \text{and} \quad Z = \{y \in \mathcal{M}^{\infty}(\mathbf{L}) \mid 0 \leq y\}.$$[3]

Our next step is to characterize the collection of frames $F = (X, Y, \preccurlyeq, R, U, Z)$ satisfying $\mathcal{G}(F) \in \mathbf{CL}$ (correspondence). In the remainder of this section, we assume that any element named x (resp. y) with any super- or subscript comes from X (resp. Y).

[3] We could also have described Z as a subset of X, however as it occurs in the axiom $(a \to 0) \to 0 = a$ and the implication is meet-preserving in the second coordinate, it is more convenient to describe it by the collection of meet-irreducibles above it, *i.e.* by a subset of Y.

By Corollary 6.14 in [4], the fusion in $\mathcal{G}(F)$ is associative iff F satisfies Φ_a:

$$\begin{aligned}&\forall x_1, x_2, x_3 \quad \forall y\\ &\quad([\forall x_2'(\forall y'[R(x_2, x_3, y') \Rightarrow x_2' \leq y'] \Rightarrow R(x_1, x_2', y))]\\ &\qquad\qquad \Leftrightarrow [\forall x_1'(\forall y''[(R(x_1, x_2, y'') \Rightarrow x_1' \leq y'')] \Rightarrow R(x_1', x_3, y))])\end{aligned}$$

Furthermore, by Corollary 6.17 in [4], the fusion in $\mathcal{G}(F)$ is commutative iff F satisfies Φ_c:

$$\forall x_1, x_2\ \forall y \ \ (R(x_1, x_2, y) \quad \Leftrightarrow \quad R(x_2, x_1, y))$$

For U to be the unit of the fusion in $\mathcal{G}(F)$ we have to ensure that $W \otimes U = W$ for all $W \in \mathcal{G}(F)$. As the fusion on $\mathcal{G}(F)$ is completely join preserving, it suffices to ensure $x \otimes U = x$ for all $x \in X(= J^\infty(\mathcal{G}(F)))$. Note that,

$$\begin{aligned} x \otimes U \leq y \quad &\Leftrightarrow \quad \bigvee\{x \otimes x' \mid x' \leq U\} \leq y\\ &\Leftrightarrow \quad \forall x' \in U.\, x \otimes x' \leq y\\ &\Leftrightarrow \quad \forall x' \in U.\, R[x, x', \_]^l \subseteq \{y\}^l\\ &\Leftrightarrow \quad \forall x' \in U.\, y \in R[x, x', \_]^{lu} = R[x, x', \_] \quad \text{(as } R \text{ is compatible)}\\ &\Leftrightarrow \quad U \subseteq R[x, \_, y]. \end{aligned}$$

Hence, U is the unit of the fusion in $\mathcal{G}(F)$ iff F satisfies Φ_u:

$$\forall x\ \forall y. \quad x \preccurlyeq y \quad \Leftrightarrow \quad U \subseteq R[x, \_, y].$$

Now we have come to the last axiom: $(a \to 0) \to 0 = a$. First note that, by the adjunction property,

$$a \leq (a \to 0) \to 0 \quad \Leftrightarrow \quad (a \to 0) \otimes a \leq 0 \quad \Leftrightarrow \quad a \to 0 \leq a \to 0.$$

So in any case $a \leq (a \to 0) \to 0$. Furthermore, the mapping $a \mapsto (a \to 0) \to 0$ is completely join preserving and therefore it again suffices to consider completely join-irreducible elements. Note that, for $x' \in \mathcal{J}^\infty(\mathcal{G}(\mathbf{F}))$,

$$\begin{aligned} x' \leq (x \to 0) \to 0 \quad &\Leftrightarrow \quad (x \to 0) \otimes x' \leq 0\\ &\Leftrightarrow \quad x \to 0 \leq x' \to 0\\ &\Leftrightarrow \quad \forall x''.\ x'' \leq x \to 0 \ \Rightarrow\ x'' \leq x' \to 0\\ &\Leftrightarrow \quad \forall x''.\ x \otimes x'' \leq 0 \ \Rightarrow\ x' \otimes x'' \leq 0\\ &\Leftrightarrow \quad \forall x''.\ Z \subseteq R[x, x'', \_] \ \Rightarrow\ Z \subseteq R[x', x'', \_] \end{aligned}$$

Hence, the equation $(a \to 0) \to 0 = a$ holds in $\mathcal{G}(F)$ iff F satisfies Φ_{dd}:

$$\forall x, x'\ (\forall x''.\ Z \subseteq R[x, x'', \_] \ \Rightarrow\ Z \subseteq R[x', x'', \_]) \ \Rightarrow\ x' \leq x.$$ [4]

Theorem 10 *The class of extended RS-frames $F = (X, Y, \preccurlyeq, R, U, Z)$ satisfying Φ_a, Φ_c, Φ_u and Φ_{dd} gives complete semantics for MALL.*

[4] Note that the statement $x' \leq x$ uses the ordering of $\mathcal{G}(F)$. We may also write this in the language of the frame as: $\forall y.\ x \preccurlyeq y \ \Rightarrow\ x' \preccurlyeq y$.

Up to now we have computed the conditions on the relational frames corresponding to the axioms in a mechanical way, not worrying about getting the simplest possible formulation. As our axioms could all be reduced to statements concerning join-irreducibles elements, these mechanical translations yield first order statements on the dual. This illustrates the strength of using duality theory in the search for relational semantics: it allows a modular and uniform treatment of additional operations and axioms. In the next section we will see that we may rewrite the conditions to get a cleaner representation and we will show that the semantics are closely related to phase spaces which are traditionally used as semantics for (multiplicative additive) linear logic.

5 Relational frames and phase semantics

The traditional models used for MALL are so-called phase spaces. A *phase space* is a tuple $(M,\cdot,1,\perp)$ where $(M,\cdot,1)$ is a commutative monoid and $\perp\subseteq M$. One defines an operation on subsets A of M by

$$A^{\perp}=\{m\,|\,\forall n\in A.\ m\cdot n\in\perp\}. \tag{3}$$

A *fact* is a subset $F\subseteq M$ such that $(F^{\perp})^{\perp}=F$. MALL is interpreted in phase spaces by assigning facts to the basic propositions and interpreting the connectives as operations on facts [8]. As, for $A,B\in\wp(M)$, $B\subseteq A^{\perp}\Leftrightarrow A\subseteq B^{\perp}$, the mapping $(\_)^{\perp}$ yields a Galois connection on $\wp(M)$ and the Galois closed sets are exactly the facts. The operations on facts corresponding to the connectives of MALL turn this collection of facts into a CL-algebra $\mathcal{F}ct(M)$. An inequality of MALL-formulas holds in a phase space M iff it holds in the corresponding CL-algebra $\mathcal{F}ct(M)$.

We will now see how phase semantics relate to the semantics derived in the previous section.

Proposition 11 *Let* $\mathbf{L}$ *be a perfect CL-algebra. The subposets* $\mathcal{J}^{\infty}(\mathbf{L})$ *and* $\mathcal{M}^{\infty}(\mathbf{L})$ *of* $\mathbf{L}$ *are dually order isomorphic.*

Proof We will show that $(\_)^{\perp}$ restricts to a map $\mathcal{J}^{\infty}(\mathbf{L})\to\mathcal{M}^{\infty}(\mathbf{L})$. The claim then follows from the fact that this map is idempotent and order-reversing. Let $x\in\mathcal{J}^{\infty}(\mathbf{L})$ and $A\subseteq\mathbf{L}$ such that $x^{\perp}=\bigwedge A$. Then

$$x=(x^{\perp})^{\perp}=(\bigwedge A)^{\perp}=\bigvee\{a^{\perp}\,|\,a\in A\}.$$

As $x\in\mathcal{J}^{\infty}(\mathbf{L})$, there exists $a\in A$ s.t. $x=a^{\perp}$, whence $x^{\perp}=(a^{\perp})^{\perp}=a$. □

By the previous proposition, for a CL-algebra $\mathbf{L}$, its completely join-irreducibles and its completely meet-irreducibles are dually order-isomorphic and therefore the algebra may be described by a one-sorted frame based on the set $\mathcal{J}^{\infty}(\mathbf{L})$. Note that, for $x_1,x_2\in\mathcal{J}^{\infty}(\mathbf{L})$,

$$x_1\leq x_2^{\perp}\ \Leftrightarrow\ x_1\leq x_2\to 0\ \Leftrightarrow\ x_1\otimes x_2\leq 0.$$

Hence, the order relation between $\mathcal{J}^\infty(\mathbf{L})$ and $\mathcal{M}^\infty(\mathbf{L})$ is completely determined by the fusion and the constant 0. Furthermore, in any CL-algebra, $1 = 0^\perp$, hence 1 is definable from 0 and the linear negation.

For a CL-algebra $\mathbf{L}$ we define a (one-sorted) frame $\mathcal{F}_1(\mathbf{L}) = (X, R_\downarrow, Z_\downarrow)$, by $X = \mathcal{J}^\infty(\mathbf{L})$, $Z_\downarrow = \{x \in X \mid x \leq 0\}$ and, for $x_1, x_2, x_3 \in X$,

$$R_\downarrow(x_1, x_2, x_3) \ \Leftrightarrow \ x_3 \leq x_1 \otimes x_2.$$

Conversely, for an RS-frame[5] $P = (X, R_\downarrow, Z_\downarrow)$ we define a Galois connection on $\wp(X)$ by, for $A \in \wp(X)$,

$$A^\perp = \{x \in X \mid \forall a \in A.\ R_\downarrow[x, a, \_] \subseteq Z_\downarrow\}. \tag{4}$$

We define a fusion on $\mathcal{G}_1(P)$, the Galois closed subsets of P, by

$$\begin{array}{rcll} x_1 \otimes x_2 & = & \bigvee R[x_1, x_2, \_] & \text{for all}\, x_1, x_2 \in X, \\ w_1 \otimes w_2 & = & \bigvee \{x_1 \otimes x_2 \mid x_1 \leq w_1, x_2 \leq w_2\} & \text{for all}\, w_1, w_2 \in \mathcal{G}_1(P). \end{array}$$

For a CL-algebra $\mathbf{L}$, the structures $\mathcal{F}(\mathbf{L}) = (X, Y, \preccurlyeq, R, U, Z)$ and $\mathcal{F}_1(\mathbf{L}) = (X, R_\downarrow, Z_\downarrow)$ are directely interdefinable. For example, for $x_1, x_2, x_3 \in X$,

$$\begin{array}{rcl} R_\downarrow(x_1, x_2, x_3) & \Leftrightarrow & \forall y \in Y.\ R[x_1, x_2, y] \ \Rightarrow \ x_3 \leq y \\ & \Leftrightarrow & x_3 \in R[x_1, x_2, \_]^l. \end{array}$$

This allows us to translate the conditions Φ_a, Φ_c, Φ_u and Φ_{dd} to statements about one-sorted frames. *E.g.*, Φ_{dd} becomes the statement Φ'_{dd}:

$$\forall x, x'\ (\forall x''.\ R_\downarrow[x, x'', \_] \subseteq Z_\downarrow \ \Rightarrow \ R_\downarrow[x', x'', \_] \subseteq Z_\downarrow) \ \Rightarrow \ x' \leq x.$$

Translation of the other statements is left to the reader. For a one-sorted RS frame P, the algebra $\mathcal{G}_1(P)$, with constants 1 and 0 defined in the evident way, is a CL-algebra iff P satisfies Φ'_a, Φ'_c, Φ'_u and Φ'_{dd}.

Theorem 12 *One sorted RS-frames $(X, R_\downarrow, Z_\downarrow)$, satisfying Φ'_a, Φ'_c, Φ'_u and Φ'_{dd} give complete semantics for MALL. We will call these structures CL-frames.*

For a CL-frame $P = (X, R_\downarrow, Z_\downarrow)$ we may define a phase space (only lacking a unit for the multiplication[6]) by $M_P = \wp(X)$, $\perp_P = \downarrow Z_\downarrow = \{A \in \wp(X) \mid A \subseteq Z_\downarrow\}$ and, for all $A, B \in \wp(X)$,

$$A \cdot_P B = \bigcup \{R[a, b, \_] \mid a \in A, b \in B\}.$$

As P satisfies Φ'_c, $\cdot_P$ is commutative.

[5] The notions 'reduced' and 'separating' are defined for one-sorted frames, as in Section 2 for two sorted frames, in such a way that they ensure that X embeds in $\mathcal{G}_1(F)$ as its completely join-irreducibles.

[6] This is not a big issue as 1 is definable from the linear negation and 0.

Lemma 13 *For all $\mathcal{A} \in \wp(\wp(X))$, if $\mathcal{A}$ is a fact, i.e. $(\mathcal{A}^\perp)^\perp = \mathcal{A}$, then $\mathcal{A}$ is a principal downset in $\wp(\wp(X))$. Furthermore, for all $A \in \wp(X)$, A is Galois closed in P iff $\downarrow A$ is a fact in M_P.*

Proof We denote both the map (3) on $\wp(M_P)$ and the map (4) on $\wp(X)$ by $(\_)^\perp$, as the reader may derive the intended meaning from the context. Note that, for $\mathcal{A} \in \wp(M_P)$,

$$\begin{aligned}
\mathcal{A}^\perp &= \{B \in M_P \,|\, \forall A \in \mathcal{A}.\ B \cdot_P A \in \perp_P\} \\
&= \{B \in M_P \,|\, \forall A \in \mathcal{A}.\ B \cdot_P A \subseteq Z_\downarrow\} \\
&= \{B \in M_P \,|\, B \cdot_P \textstyle\bigcup \mathcal{A} \subseteq Z_\downarrow\} \\
&= \{B \in M_P \,|\, B \subseteq (\textstyle\bigcup \mathcal{A})^\perp\} \\
&= \downarrow((\textstyle\bigcup \mathcal{A})^\perp).
\end{aligned}$$

From which the first claim follows immediately. The second claim easily follows from $((\downarrow A)^\perp)^\perp = \downarrow((A^\perp)^\perp)$.

Theorem 14 *The CL-algebras $\mathcal{G}_1(P)$ and $\mathcal{F}ct(M_P)$ are isomorphic.*

Proof It follows from the previous lemma that the mapping $A \mapsto \downarrow A$ is a bijection between the two underlying sets. It is left to the reader to check that this map preserves the CL-structure.

Using the previous theorem, completeness of the semantics of phase spaces may be derived from completeness of CL-frames. It is not always possible to construct, given a phase space M, a CL-frame P_M s.t. $\mathcal{G}_1(P_M) \cong \mathcal{F}ct(M)$, as the complete lattice $\mathcal{F}ct(M)$ may not be perfect.

The main advantage of working with CL-frames, in stead of phase spaces, is that they are in a duality with perfect CL-algebras which enables a modular and uniform treatment of additional axioms and operations. Furthermore, the phase space describing a specific CL-algebra (e.g. the Lindenbaum algebra used in the completeness proof) is in general much larger than the corresponding CL-frame. This size difference is also visible in the proof of Theorem 14: the underlying set of the phase space associated to a CL-frame $(X, R_\downarrow, Z_\downarrow)$ is $\wp(X)$.

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